RESTRICTIONS ON HILBERT COEFFICIENTS GIVE THE DEPTH OF A PRIME IDEAL INSIDE THE POLYNOMIAL RING

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Abstract. In this paper, we prove that for a prime ideal P of dimension r inside a polynomial ring, if adjoining s general linear forms to the prime ideal changes the $r-s$-th Hilbert coefficient by 1 and doesn't change the 0th to $r-s-1$-th Hilbert coefficients where $s \leq r$, then the depth of S/P is $n-s-1$. This criteria also tells us about possible restrictions on the generic initial ideal of a prime ideal inside a polynomial ring.

1. Introduction

Let k be an infinite field. The famous Eisenbud-Goto conjecture claims that an inequality holds between several numerical invariants of a homogeneous prime P in a polynomial ring over k:

Conjecture 1.1 ([4]). *Let k be an algebraically closed field, $P \subset (x_1, \dots, x_n)^2$ be a homogeneous ideal in $S = k[x_1, \dots, x_n]$, then*

$$\operatorname{reg}(P) \leq \deg(S/P) - \operatorname{ht}(P) + 1$$

Here $\deg(S/P)$ is the multiplicity of S/P. The conjecture is proved for many special cases including curves and smooth surfaces but it is false in general. The first counterexample is given by McCullough and Peeva in [9] using Rees-like algebras. It means that the regularity can be quite large in general even when the ring is a graded domain.

Let P be a homogeneous prime ideal of S and $<=<_{rev}$ be the graded reverse lexicographic order on S, then we can talk about the generic initial ideal $\operatorname{gin}_<(P)$ of P. The above conjecture involves several invariants of P including multiplicity, regularity and height, and they can all be described using generic initial ideal in a simple way. Let $J = \operatorname{gin}_<(P)$ and $G(J)$ be the set of monomial minimal generators of J. Then by Bayer and Stillman's theorem in [1] and Eliahou and Kervaire's theorem in [5], we know: $\operatorname{reg}(S/P) = \max\{\deg(u) : u \in G(J)\} - 1$, and $\operatorname{depth}(S/P) = n - \max\{i : x_i | u \in G(J)\}$ so a description of such generic initial ideal may lead to similar inequalities of these invariants. Therefore, it makes sense to study what are possible generic initial ideals of primes in a polynomial ring. Although we have a description on all the possible generic initial ideals (for instance, see [7]) of an ideal, there may be more strict restrictions on the generic initial ideal of a homogeneous prime ideal. This paper gives such a restriction and shows that certain monomial ideals are not the initial ideal of a homogeneous prime ideal. Moreover, we describe the restriction using Hilbert coefficients of the ring S/P which leads to S/P being almost Cohen-Macaulay.

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2. Preliminaires

Before we describe the restriction on generic initial ideals of prime ideals, we introduce some notations on polynomial rings, monomials and monomial orders. Let $S(n) = k[x_1, \ldots, x_n]$ be a polynomial ring in n-variables, $\mathfrak{m}(n)$ be its homogeneous maximal ideal. If $n_1 < n_2$, there is a natural embedding $\eta : S(n_1) \hookrightarrow S(n_2)$. If a linear form $l = c_1x_1 + \ldots + c_nx_n \in S(n)_1$ satisfies $c_n \neq 0$, then the map $\eta' : S(n-1) \xrightarrow{\eta} S(n) \to S(n)/lS(n)$ is an isomorphism. Denote the map $\pi_l : S(n) \to S(n)/lS(n) \xrightarrow{(\eta')^{-1}} S(n-1)$. If $l = x_n$, we denote $\pi_n = \pi_{x_n} : S(n) \to S(n-1)$. For a sequence of linear forms $l_1, \ldots, l_s$ we define the map iterately:

$$\Pi_{l_1,\ldots,l_s} = \pi_{\overline{l_s}}\pi_{\overline{l_{s-1}}} \ldots \pi_{l_1} : S(n) \to S(n-s)$$

where $\overline{l_i}$ is the image of l_i under $\Pi_{l_1,\ldots,l_{i-1}}$. This definition does not make sense for all sequences of linear forms; when it make sense, the x_n-coefficient of l_1 is nonzero, the x_{n-1}-coefficient of $\bar{l_2} = \pi_{l_1}(l_2)$ is nonzero, and so on. The above conditions are finitely many conditions given by the nonvanishing of certain polynomials in terms of the coefficients of $l_1, \ldots, l_s$. So $\Pi_{l_1,\ldots,l_s}$ makes sense on a Zariski open set in the space of all s linear forms. In particular they make sense for s general linear forms. Besides, they also make sense when $s = n - d$ and $l_1 = x_n, l_2 = x_{n-1}, \ldots, l_s = x_{n-d+1}$. In this case we define

$$\Pi_d = \Pi_{x_n,\ldots,x_{n-d+1}} = \pi_{d+1}\pi_{d+2} \ldots \pi_n : S(n) \to S(d)$$

for $1 \leq d \leq n-1$. We denote $\Pi_n = \mathrm{id}_{S(n)}$. If $I \subset S(n)$ is a homogeneous $S(n)$-ideal, define the saturation of I to be $I^{sat} = I : \mathfrak{m}(n)^\infty$.

We recall the definition of initial ideal and generic initial ideal taken from [6] and [7]. Let $<$ be a monomial order on $S(n)$. For $f \in S(n)$, we can write f as a k-linear combination of monomials, that is, $f = \sum_u a_u u, a_u \in k$. Define the initial of f with respect to $<$, denoted by $\mathrm{in}_<(f)$, to be the largest monomial u such that $a_u \neq 0$. The initial ideal of I is:

Definition 2.1. $\mathrm{in}_<(I) = (\mathrm{in}(f) | f \in I)$.

Now suppose k is an infinite field. Every linear map $\alpha \in GL_n(k)$ defines a linear automorphism of $S(n)$. The generic initial ideal of I is defined as follows:

Definition 2.2. There exist a Zariski open set U of $GL_n(k)$ such that for all $\alpha \in U$, $\mathrm{in}_<(\alpha(I))$ is independent of α. This common initial ideal is called the generic initial ideal of I, denoted by $\mathrm{gin}_<(I)$ or $\mathrm{gin}(I)$ if the order is clear.

Throughout this paper, we will use the graded reverse lexicographic order $<=<_{rev}$.

Definition 2.3. We say a monomial ideal J is of Borel type if for any $i < j$, a monomial $u \in J$ dividing x_j, then there is some integer s such that $x_i^s u/x_j \in J$.

Equivalently, this is saying $J : (x_1, \ldots, x_i)^\infty = J : x_i^\infty$ for any i.

Proposition 2.4. *For any homogeneous ideal I, $\mathrm{gin}(I)$ is always of Borel type. Moreover, $\dim(S) - \dim(S/I) = d$ if and only if $\mathrm{gin}(I)$ contains pure powers of $x_1, \ldots, x_d$ but not pure powers of $x_{d+1}, \ldots, x_n$.*

We also recall the definition of Hilbert coefficients taken from [3]. Let M be a finitely generated graded $S(n)$-module. The function $H_M : \mathbb{N} \to \mathbb{N}, H_M(n) = \dim_k(M_n)$ is called the Hilbert function of M and the power series $h_M(t) =$

$\Sigma_{i\in\mathbb{N}} H_M(i)t^i$ is called the Hilbert series. It is well known that the Hilbert series is of the form $q_M(t)/(1-t)^d$ with $d = \dim(M)$, $q_M(t)$ is a polynomial with integer coefficients satisfying $q_M(1) \neq 0$.

Definition 2.5. Let M be a finitely generated graded $S(n)$-module of dimension d with Hilbert series $q_M(t)/(1-t)^d$. Expand $q_M(t)$ as linear combinations of powers of $t-1$:

$$q_M(t) = e_0 + e_1(t-1) + e_2(t-1)^2 + \dots$$

The coefficient e_i is called the i-th Hilbert coefficient of M.

Since $q_M(t)$ has integer coefficients, all the Hilbert coefficients are integers.

3. Generic initial ideal under projection and saturation

For a monomial $u = x_1^{e_1}x_2^{e_2}\dots x_n^{e_n}$, we denote $\phi_i(u) = u/x_i^{e_i}$, that is, we eliminate all x_i's from the factors of u. Denote $\Phi_d(u) = \phi_{d+1}\phi_{d+2}\dots\phi_n(u)$. If $J = (u_1, \dots, u_s)$ is a monomial ideal in $S(n)$ with monomial minimal generator $u_1, \dots, u_s$, We denote $\phi_i(J) = (\phi_i(u_1), \dots, \phi_i(u_s)) = J : {x_i}^{\infty}$, $\Phi_d(J) = (\Phi_d(u_1), \dots, \Phi_d(u_s)) = J : (x_{d+1}\dots x_n)^{\infty}$, $\overline{\Phi_d}(J) = \pi_{d+1}\phi_{d+1}\pi_{d+2}\phi_{d+2}\dots\pi_n\phi_n(J)$. Note that $\Phi_d(J)$ is an $S(n)$-ideal while $\overline{\Phi_d(J)}$ is an $S(d)$ ideal, and these two ideals are generated by the same set of monomials in $S(d)$.

Remark 3.1. If $i \neq j$, then $\pi_i\phi_j = \phi_j\pi_i$. To be precise, if J is generated by $u_1, \dots, u_s$, then $\pi_i\phi_j(J) = \phi_j\pi_i(J)$ is generated by $\phi_j(u_k)$ where u_k is not divisible by x_i. Thus in the expression of $\overline{\Phi_d}(J) = \pi_{d+1}\phi_{d+1}\pi_{d+2}\phi_{d+2}\dots\pi_n\phi_n(J)$, we can commute all ϕ's and π's if we only move π to the left. This implies $\overline{\Phi_d}(J) = \pi_{d+1}\pi_{d+2}\dots\pi_n\phi_{d+1}\phi_{d+2}\dots\phi_n(J) = \Pi_d\Phi_d(J)$.

The following two properties shows that the generic initial ideal in reverse lexicographic order behaves well under the projection map and saturation:

Proposition 3.2 ([6], Proposition 2.14)**.** *Let I be a homogeneous ideal in $S(n)$, l be a general linear form in $S(n)$. Then* $\mathrm{gin}(\pi_l(I)) = \pi_n(\mathrm{gin}(I))$.

Remark 3.3. Here $\pi_l(I)$ is an ideal of $S(n-1)$ so the generic initial ideal is well-defined. $\pi_n(\mathrm{gin}(I))$ is also an ideal of $S(n-1)$, thus this equality makes sense because it compares two ideals in the same ring $S(n-1)$.

Proposition 3.4. $\mathrm{gin}(I^{sat}) = \mathrm{gin}(I) : x_n^{\infty} = \mathrm{gin}(I)^{sat}$.

Proof. The first equality is proved in [6], Proposition 2.21. The second statement is true because $\mathrm{gin}(I)$ is of Borel type. □

Let I be a saturated homogeneous ideal in $S(n)$. Let l be a linear form such that the x_n-coefficient of l is nonzero. We call the ideal $\widetilde{\pi_l}(I) = (\pi_l(I))^{sat}$ the section with one hyperplane. It is a saturated ideal in $S(n-1)$. If we have s linear forms $l_1, \dots, l_s$ such that $\Pi_{l_1,\dots,l_s}$ is well-defined, then inductively we can define the ideal of section with s hyperplanes: the ideal of section with one hyperplane is $I_1 = \widetilde{\pi_{l_1}}(I)$, it is an ideal in $S(n-1)$; let $\overline{l_2}$ be the image of l_2 under π_{l_1}, so define the section with two hyperplanes $I_2 = \pi_{\overline{l_2}}(I_1)^{sat}$, and inductively, with s hyperplanes is $I_s = \pi_{\overline{l_s}}(I_{s-1})^{sat}$. Let $d = n-s$, so I_s is an ideal in $S(d)$. In algebraic geometry, we can consider the algebraic set X of $\mathbb{P}^{n-1}$ corresponding to I; the intersection of X with s hyperplanes defined by $l_1, \dots, l_s$ is X_s. Then

I_s will be the defining ideal of X_s inside the linear subvariety $\mathbb{P}^{n-s-1}$ defined by $l_1, \ldots, l_s$. The coordinate ring of the linear subvariety is $S(n)/(l_1, \ldots, l_s)$ which can be identified with $S(n-s) = S(d)$.

Proposition 3.5. $\mathrm{gin}(I_s) = \widetilde{\pi_{d+1}}\widetilde{\pi_{d+2}} \ldots \widetilde{\pi_n}(\mathrm{gin}(I))$.

Proof. Apply proposition 3.2 and proposition 3.4 inductively. □

Lemma 3.6. *Let J be a monomial ideal of Borel type and $1 \le d \le n$. Then $\Pi_d(J)$, $\Phi_d(J)$ and $\overline{\Phi_d(J)}$ are all of Borel type.*

Proof. Suppose $J = (u_1, \ldots, u_s)$, then $\Phi_d(J) = (\Phi_d(u_1), \ldots, \Phi_d(u_s))$. Every minimal generator of $\Phi_d(J)$ is of the form $\Phi_d(u_k)$ for some k. Choose i such that $x_i|\Phi_d(u_k)$ and choose $j < i$. Then $i \le d$ because $\Phi_d(u_k)$ is only divisible by a subset of $\{x_1, \ldots, x_d\}$. In this case $x_i|u_k$. Since J is of Borel type, there exist t such that ${x_j}^t u_k/x_i$ is in J, so there is another minimal generator $u_{k'}$ of J with $u_{k'}|{x_j}^t u_k/x_i$. Since $j < i \le d$,$\Phi_d(u_{k'})|\Phi_d({x_j}^t u_k/x_i) = {x_j}^t \Phi_d(u_k)/x_i$. This means that $\Phi_d(J)$ is still of Borel type. For Π_d, we have $\Pi_d(J) = (\Pi_d(u_1), \ldots, \Pi_d(u_s))$, and the set of minimal generators of $\Pi_d(J)$ is just the set of $\Pi_d(u_k) = u_k$'s where $\Pi_d(u_k) \neq 0$. Choose i such that $x_i|\Pi_d(u_k)$ and choose $j < i$. Then $i \le d$ because $\Pi_d(u_k) \neq 0$ is only divisible by a subset of $\{x_1, \ldots, x_d\}$. Then we can use the same argument as $\Phi_d(J)$ to prove $\Pi_d(J)$ is of Borel type. The last statement is true by the previous two statements because $\overline{\Phi_d} = \Pi_d \Phi_d$. □

Proposition 3.7. *Let J be a saturated monomial ideal of Borel type of $S(n)$. Then*

(1) $\pi_{d+1}\widetilde{\pi_{d+2}} \ldots \widetilde{\pi_n}(J) = \pi_{d+1}\phi_{d+1} \ldots \pi_n\phi_n(J) = \overline{\Phi_d(J)}$.
(2) $\widetilde{\pi_{d+1}}\widetilde{\pi_{d+2}} \ldots \widetilde{\pi_n}(J) = \phi_d\pi_{d+1}\phi_{d+1} \ldots \pi_n\phi_n(J) = \phi_d\overline{\Phi_d(J)}$.

Proof. J is of Borel type and saturated, so $J = J^{sat} = J : {x_n}^\infty = \phi_n(J)$ and $\pi_n(J) = \pi_n\phi_n(J)$, which means (1) is true for $d = n-1$ and (2) is true for $d = n$. It is obvious that (2) is true for d implies (1) for $d-1$, so by induction it suffices to show (1) for d implies (2) for d. By Lemma before we see $\overline{\Phi_d(J)}$ is of Borel type, so if (1) is true for d, then $\widetilde{\pi_{d+1}}\widetilde{\pi_{d+2}} \ldots \widetilde{\pi_n}(J) = (\pi_{d+1}\widetilde{\pi_{d+2}} \ldots \widetilde{\pi_n}(J))^{sat} = (\overline{\Phi_d(J)})^{sat} = \phi_d\overline{\Phi_d(J)}$, so (2) is true for d and we are done. □

4. The main theorem

This section describes restrictions on the generic initial ideal of a homogeneous prime inside $S(n)$.

Theorem 4.1. *Let P be a homogeneous prime ideal in $S(n)$ and $J = \mathrm{in}(P)$ be the initial ideal of P. Assume J is of Borel type, and for some $1 \le d \le n-1$ we have $\overline{\Phi_d}(J) = \Pi_d(J) + u$ for some $u \in \Pi_d(J) : \mathfrak{m}(d)$. Then:*

(1) *Either $u = 1$, $(x_1, \ldots, x_d) \subset J$, or J is generated by $J \cap k[x_1, \ldots, x_d]$ and one extra generator v.*
(2) *If $u \neq 1$, then the extra generator $v = ux_{d+1}^e$ for some $e > 0$.*
(3) *If $u \neq 1$, then J is generated by monomials inside $k[x_1, \ldots, x_{d+1}]$.*

Proof. Let $f \in P$ such that $\mathrm{in}(f)$ is a minimal generator of J. We claim f is an irreducible polynomial. If f is not irreducible, write $f = f_1f_2$, f_1, f_2 are not constants. Then $\mathrm{in}(f) = \mathrm{in}(f_1)\mathrm{in}(f_2)$ with $\mathrm{in}(f) \neq \mathrm{in}(f_1)$ or $\mathrm{in}(f_2)$. Since P is prime and $f \in P$, $f_1 \in P$ or $f_2 \in P$, which implies $\mathrm{in}(f_1) \in J$ or $\mathrm{in}(f_2) \in J$, which contradicts the minimality of $\mathrm{in}(f)$. Let $u_1, \ldots, u_s, v_1, \ldots, v_t$ be the monomial minimal generators of J with $u_i \in k[x_1, \dots, x_d]$, $v_j \notin k[x_1, \dots, x_d]$. Since $\overline{\Phi_d}(J) \neq \Pi_d(J)$, one of the generator does not lie in $k[x_1, \dots, x_d]$, that is, $t \geq 1$ and there is at least one v_1. Then $\Phi_d(u_i) = u_i$, $\Phi_d(v_j) \neq v_j$. $\Phi_d(J)$ is generated by $\Phi_d(u_i) = u_i$ and $\Phi_d(v_j)$, and is minimally generated by u_i, u. $\overline{\Phi_d(J)}$ is also generated by u_i, u, viewed as a monomial ideal in $S(d)$. For any i, j, u_i does not divide v_j; but $\Phi_d(v_j)$ divides v_j, so u_i does not divide $\Phi_d(v_j)$. But $\Phi_d(v_j) \in \Phi_d(J) = (u_1, \dots, u_s, u)$, so $\Phi_d(v_j) \in (u_1, \dots, u_s, u) \backslash (u_1, \dots, u_s)$; this forces $\Phi_d(v_j) = u$ because $u \in \Pi_d(J) : \mathfrak{m}(d) = (u_1, \dots, u_s) : \mathfrak{m}(d)$. Write $v_j = uw_j$ for $w_j \in k[x_{d+1}, \dots, x_n]$. We have seen $t \geq 1$. If $t = 1$ this is the second part of (1). Suppose $t \geq 2$, we claim the first part of (1) holds. Assume $v_1 = uw_1$, $v_2 = uw_2$ are two distinct monomials minimal generator of J. Take f_1, f_2 to be the elements in the reduced Gröbner basis of P satisfying $\mathrm{in}(f_1) = v_1, \mathrm{in}(f_2) = v_2$. Every monomial appearing in f_1, f_2 is not in J except their initials. By the claim at the beginning of the proof, f_1, f_2 are irreducible. We can write $f_1 = up_1 + q_1, f_2 = up_2 + q_2$ such that up_1 is the sum of all terms appearing in f_1 divisible by u, and q_1 is the sum of all other terms; similar for p_2, q_2. Then $\mathrm{in}(p_1) = w_1, \mathrm{in}(p_2) = w_2$. Besides, $u \in (u_1, \dots, u_s) : (x_1, \dots, x_d)$, so any other monomial m appearing in q_1 are not divisible by $x_1, \dots, x_d$, otherwise the mu divides some u_i, so it lies in J which is a contradiction. This means $p_1 \in k[x_{d+1}, \dots, x_n]$. Similarly $p_2 \in k[x_{d+1}, \dots, x_n]$. Consider the polynomial $F = p_2 f_1 - p_1 f_2 = p_2 q_1 - p_1 q_2$. Take any term m_1 of p_2 and m_2 of q_1. Then $m_1 \in k[x_{d+1}, \dots, x_n]$. Also, m_2 is not divisible by u_i since the terms of q_1 appears in f_1, and is not divisible by u by construction of q_1. This means $m_2 \notin (u_1, \dots, u_s, u) = \Phi_d(J) = J : (x_{d+1} \dots x_n)^\infty$. So $m_1 m_2 \notin J$. This is true for any choice of $m_1 m_2$. Similarly the product of any term of p_1 and any term of q_2 does not lie in J. So any possible term of F does not lie in $J = \mathrm{in}(P)$. But $F \in P$, so $F = 0$ and $p_2 f_1 = p_1 f_2$. Since f_1 is irreducible we have $f_1 | p_1$ or $f_1 | f_2$. If $f_1 | p_1$ then $\mathrm{in}(f) = v_1 = uw_1 | w_1 = \mathrm{in}(p_1)$. This means f_1 and p_1 differ by a constant and $u = 1 \in \Pi_d(J) : \mathfrak{m}(d)$, so J contains $(x_1, \dots, x_d)$, and the first part of (1) holds. Similarly if $f_2 | p_2$, the first part of (1) also holds. If both of the above is false, then f_1 and f_2 differ by a constant, which means $v_1 = \mathrm{in}(f_1) = \mathrm{in}(f_2) = v_2$, which is contradictory to $v_1 \neq v_2$. Hence (1) is proved. Suppose (2) is false, and $v = uw \in J$ where w is not a pure power of x_{d+1}. Then $x_j | w$ for some $j > d + 1$. Since J is of Borel type, there exist $e > 0$ such that $v' = uwx_{d+1}^e / x_j \in J$. Pick a monomial minimal generator v" dividing u'. v" does not divide v by minimality and v does not divies v" by construction. Then $v = uw, v" = uw"$, $w, w" \in k[x_{d+1}, \dots, x_n]$, and $w, w"$ does not divide each other, which is contradictory to the second part of (1). (3) is a corollary of (1) and (2). □

Corollary 4.2. *Let P be a homogeneous prime ideal in $S(n)$ and $J = \mathrm{gin}(P)$ be the initial ideal of P. Assume for some $1 \leq d \leq n - 1$ we have $\overline{\Phi_d}(J) = \Pi_d(J) + u$ for some $u \in \Pi_d(J) : \mathfrak{m}(d)$. Then:*

1. *Either $u = 1$, $(x_1, \dots, x_d) \subset J$, or J is generated by $J \cap k[x_1, \dots, x_d]$ and one extra generator v.*
2. *If $u \neq 1$, then the extra generator $v = ux_{d+1}^e$ for some $e > 0$.*
3. *If $u \neq 1$, then J is generated by monomials inside $k[x_1, \dots, x_{d+1}]$.*

Proof. The generic initial ideal of any ideal is always of Borel type. By definition of the generic initial ideal, $J = \mathrm{in}(\alpha(P))$ for some $\alpha \in GL_n(k)$ and $\alpha(P)$ is still a prime ideal in $S(n)$, so the conclusion follows from theorem 4.1. □

Example 4.3. The above theorem and corollary rule out some monomial ideal as the initial ideal of a prime ideal. For example, let $J = (x_1^2, x_1x_2^2, x_1x_2x_3, x_1x_3^2) \subset k[x_1, \ldots, x_n]$. Then J is of Borel type. Let $d = 1$, we see $\Phi_d(J) = (x_1)$, $\Pi_d(J) = (x_1^2)$, and $x_1 \subset (x_1^2) : (x_1)$. However, $x_1 \neq 1$ and J has a minimal generator dividing x_3. So J satisfies the condition of theorem 4.1 except that it is the initial ideal of P, and it violates the conclusion of (1), so it can not be the initial ideal or generic initial ideal of any prime ideal.

The above result can be strengthened by replacing J with a larger ideal $\Phi_{d+2}(J)$ using Bertini's irreducibility theorem. First we recall the statement of the theorem:

Theorem 4.4 ([8])**.** *Let k be an infinite field, P a homogeneous prime ideal in the polynomial ring S, s be a positive integer, $s \leq \dim(S/P) - 2$. Then for s general linear forms $l_1, \ldots, l_s$, $\widetilde{\pi_{l_s}}\widetilde{\pi_{l_{s-1}}} \ldots \widetilde{\pi_{l_1}}(P)$ is still a prime ideal in $S/(l_1, \ldots, l_s)$.*

Corollary 4.5. *Let P be a homogeneous prime ideal in $S(n)$ and $J = \mathrm{gin}(P)$. Assume for some $1 \leq d \leq n-2$ we have $\dim(S/P) \geq n-d$ and $\overline{\Phi_d}(J) = \Pi_d\Phi_{d+2}(J)+u$ for some $u \in \Pi_d\Phi_{d+2}(J) : \mathfrak{m}$. Then:*

1. *J is generated by $J \cap k[x_1, \ldots, x_d]$ and one extra generator v.*
2. *$v = ux_{d+1}^e$ for some $v \in \Pi_d(J) : \mathfrak{m}$ and $e > 0$.*
3. *J is generated by monomials inside $k[x_1, \ldots, x_{d+1}]$.*
4. $\mathrm{depth}(S/P) = n-d-1$. *If moreover* $\dim(S/P) = n-d$ *then S/P is almost Cohen-Macaulay.*

Proof. If $n = d + 2$ this is just corollary 4.2, so we may assume $n \geq d + 3$. We choose $s = n - d - 3$ general linear forms $l_1, \ldots, l_s$ and let $P_s = \widetilde{\pi_{l_s}}\widetilde{\pi_{l_{s-1}}} \ldots \widetilde{\pi_{l_1}}(P)$. If $n = d + 3$ just choose $P_s = P$. It is a prime ideal in $S(n)/(l_1, \ldots, l_s) = S(d + 3)$ because $s = n - d - 3 < \dim(S/P) - 2$. By proposition 3.7 $\mathrm{gin}(P_s) = \phi_{d+3}\overline{\Phi_{d+3}(J)} = \Pi_{d+3}\Phi_{d+2}(J)$. By Bertini's irreducibility theorem P_s is a prime ideal, so $\mathrm{gin}(P_s) = \Pi_{d+3}\Phi_{d+2}(J)$ satisfies the conclusion of (1)-(3), that is, it is generated by monomials in $x_1, \ldots, x_d$ plus an extra generator $v = ux_{d+1}^e$ where u is a monomial in $x_1, \ldots, x_d$. Since the generator of $\Phi_{d+2}(J)$ does not involve $x_{d+2}, \ldots, x_n$, the generators of $\Pi_{d+3}\Phi_{d+2}(J) \subset S(d + 3)$ and $\Phi_{d+2}(J) \subset S(n)$ are the same, and $\Phi_{d+2}(J)$ also satisfies (1)-(3) because these properties only depends on monomial generators. Now we claim J also satisfies (1)-(3). If $\Phi_{d+2}(J)$ satisfies (1)-(3) but J does not, then $\Phi_{d+2}(J) \neq J$; hence J has a minimal generator $v = w_1w_2$ where $w_1 \in k[x_1, \ldots, x_{d+1}]$, $w_2 \in k[x_{d+2}, \ldots, x_n]$, and $w_2 \neq 1$. Since J is of Borel type,there exists $e > 0$ such that $w_1x_{d+2}^e \in J$, and since $w_1x_{d+2}^e$ does not involve $x_{d+3}, \ldots, x_n$, $\Phi_{d+2}(w_1x_{d+2}^e) = w_1x_{d+2}^e \in \Phi_{d+2}(J)$. This monomial is divisible by another minimal generator v' of $\Phi_{d+2}(J)$; write $v' = w_1'w_2'$ where $w_1' \in k[x_1, \ldots, x_{d+1}]$, w_2' is a power of x_{d+2} which is not 1. This means $\Phi_{d+2}(J)$ has a minimal generator involving x_{d+2} which is contradictory to (3), so we are done. □

5. Difference in the Hilbert coefficients

The previous section talks about restrictions on $\mathrm{gin}(P)$ for a prime ideal P. However, the generic initial ideal of an ideal is hard to capture in practice as its computation requires the information of some unknown $\alpha \in GL_n(k)$. It would be easier to describe the restriction using Hilbert coefficients which is totally computable from the Hilbert function of the quotient ring. We want to see how the Hilbert coefficients change after going modulo s general linear forms.

Lemma 5.1. *Let M, N be two graded modules over some polynomial ring S of dimension r. Assume there is an exact sequence*

$$0 \to M_1 \to N \to M \to M_2 \to 0.$$

Denote $\dim M_1 = r_1, \dim M_2 = r_2, s = \max\{r_1, r_2\}$. *Then:*

1. $r \geq s$.
2. $e_i(N) = e_i(M)$ *for* $i < r - s$.
3. *If* $r_1 = s > r_2$, *then* $e_{r-s}(N) = e_{r-s}(M) + (-1)^{r-s} e_0(M_1)$.
4. *If* $r_1 < s = r_2$, *then* $e_{r-s}(N) = e_{r-s}(M) - (-1)^{r-s} e_0(M_2)$.
5. *If* $r_1 = s = r_2$, *then* $e_{r-s}(N) = e_{r-s}(M) + (-1)^{r-s} e_0(M_1) - (-1)^{r-s} e_0(M_2)$.

Proof. We have $\dim M \geq \dim M_1$ and $\dim N \geq \dim M_2$ which implies (1). For the rest statements, note that the Hilbert series is additive on short exact sequences, so $h_M(t) = h_N(t) + h_{M_2}(t) - h_{M_1}(t)$. Let $h_M(t) = q_M(t)/(1-t)^r, h_{M_1}(t) = q_{M_1}(t)/(1-t)^{r_1}, h_{M_2}(t) = q_{M_2}(t)/(1-t)^{r_2}$. Then $h_N(t) = q_N(t)/(1-t)^r$ where $q_N(t) = q_M(t) + q_{M_2}(t)(1-t)^{r-r_2} - q_{M_1}(t)(1-t)^{r-r_1}$. Now we expand both sides in terms of powers of $t-1$ and look at the coefficients of $1, (t-1), \ldots, (t-1)^{r-s}$. □

Now let J be a monomial ideal of Borel type in $S = S(n)$, $\dim(S/J) = r$, and d is an integer satisfying $d \geq n - r$. Let J_1 be the ideal generated by $J \cap k[x_1, \ldots, x_d]$ and $J_2 = \Phi_d(J)$. Then $J_1 \subset J \subset J_2$ with $\Pi_d(J_1) = \Pi_d(J)$.

Lemma 5.2. *We have:*

1. $e_i(S/J) = e_i(S/J_2)$ *for* $0 \leq i \leq r - n + d$;
2. $e_i(S/J) = e_i(S/J_1)$ *for* $0 \leq i \leq r - n + d - 1$, *and*

$$\begin{aligned} & e_{r-n+d}(S/J) - e_{r-n+d}(S/J_1) \\ &= (-1)^{r-n+d} \mathrm{rank}_{k[x_{d+1}, \ldots, x_n]}(J_2/J_1) \\ &= (-1)^{r-n+d} \dim_k \Pi_d(J_2)/\Pi_d(J_1) \end{aligned}$$

which is finite.

Proof. For any $1 \leq i \leq n-1$, $\Phi_{i+1}(J) \subset \Phi_i(J) = \Phi_{i+1}(J) : x_{i+1}^{\infty}$. For any monomial $u \in \Phi_i(J) \backslash \Phi_{i+1}(J)$, by definition there is some $e > 0$ such that $u x_{i+1}^e \in \Phi_{i+1}(J)$. Since J is of Borel type, so is $\Phi_{i+1}(J)$, so for any $j \leq i$, there exist some $e' > 0$ such that $u x_j^{e'} \in \Phi_{i+1}(J)$. This means $\Phi_i(J)/\Phi_{i+1}(J)$ is annihilated by some power of $(x_1, \ldots, x_{i+1})$, so $\dim(\Phi_i(J)/\Phi_{i+1}(J)) \leq n - i - 1$. Now consider the exact sequence

$$0 \to \Phi_i(J)/\Phi_{i+1}(J) \to S(n)/\Phi_{i+1}(J) \to S(n)/\Phi_i(J) \to 0.$$

By the argument above and lemma 5.1, we know $e_j(S/\Phi_i(J)) = e_j(S/\Phi_{i+1}(J))$ for $j < r - (n - i - 1) = r - n + i + 1$, so $e_j(S/J) = e_j(S/\Phi_d(J))$ for $j \leq r - n + d$. For J_1, by the arguement above it suffices to prove the equality if we replace J by J_2. We know S/J_1 and S/J_2 are both free $k[x_{d+1}, \ldots, x_n]$-module since J_1 and J_2 are generated by monomials in $x_1, \ldots, x_d$. Now for any monomial minimal generator u of J_2 satisfying $u \notin J_1$, we know $u \in k[x_1, \ldots, x_d]$ and there exists $s > 0$ such that $x_{d+1}^s u \in J$. So $x_i^s u \in J$ for all $1 \leq i \leq d$. By definition of J_1, $x_i^n u \in J_1$ for such i. This means that $J_2 \subset J_1 : (x_1, \ldots, x_d)^{\infty}$. So J_2/J_1 is a free $S(d)$-module of finite rank. It has dimension exactly d and $e_0(J_2/J_1) = \mathrm{rank}_{S(d)}(J_2/J_1) = \dim_k(\Pi_d(J_2)/\Pi_d(J_1))$. So apply lemma 5.1 to the exact sequence $0 \to J_2/J_1 \to S/J_1 \to S/J_2 \to 0$ we know $e_i(S/J_2) = e_i(S/J_1)$ for $0 \leq i \leq r - (n-d) - 1$ and $e_{r-n+d}(S/J_2) = e_{r-n+d}(S/J_1) + (-1)^{r-n+d} e_0(J_2/J_1)$. □

Lemma 5.3. *Let J be a monomial ideal of Borel type in $S(n)$ with $\dim(S/J) \geq n-d$. Then $\overline{\Phi_d}(J) = \Pi_d(J) + u$ for some $u \in \Pi_d(J) : \mathfrak{m}(d)$ if and only if*

$$\mathrm{rank}_{k[x_{d+1},\dots,x_n]}(J_2/J_1) = \dim_k(\Pi_d(J_2)/\Pi_d(J_1)) = 1.$$

Proof. Since J_2 and J_1 are both free $k[x_{d+1},\dots,x_n]$-modules, their quotient is a free $k[x_{d+1},\dots,x_n]$-module of rank 1 if and only if after modulo $(x_{d+1},\dots,x_n)$ the quotient is a one dimensional k-vector space, if and only if $\overline{\Phi_d}(J)/\Pi_d(J) = k$. □

Theorem 5.4. *Let P be a homogeneous prime ideal in $S(n)$, $\dim S(n)/P = r$. Take $1 \leq s \leq r$, and let $d = n-s$. choose s general linear forms $l_1,\dots,l_s$. Denote $P_1 = \pi_{l_1}\dots\pi_{l_s}(P) \subset S(d)$. Suppose $e_i(S/P) = e_i(S/P_1)$ for $1 \leq i \leq r-s-1$ and $e_{r-s}(S/P) = e_{r-s}(S/P_1) + (-1)^{r-s}$, then $\mathrm{depth}(S/P) = n-d-1$.*

Proof. Let $J = \mathrm{gin}(P)$. Denote J_1, J_2 as before. Then $J_1 = \mathrm{gin}(P_1)$ by proposition 3.5. Now taking generic initial ideal does not change the Hilbert series, so the Hilbert coefficients are the same, so we have $e_i(S/J) = e_i(S/J_1)$ for $1 \leq i \leq r-s-1$ and $e_{r-s}(S/J) = e_{r-s}(S/J_1) + (-1)^{r-s}$. Since $s = n-d$, by lemma 5.3 we know $\overline{\Phi_d}(J) = \Pi_d(J) + u$ for some $u \in \Pi_d(J) : \mathfrak{m}(d)$. Now by theorem 4.1 we know $\mathrm{depth}(S/P) = n-d-1$. □

6. Simple cases and case where $n-d = \dim(S/P)$

We can also talk about the generic initial ideal of a prime ideal when this prime is very simple.

Proposition 6.1. *Let $n \geq 3$ and J be a monomial ideal in $S(n)$. Then there exists a prime ideal P of S such that $J = \mathrm{gin}(P)$ for some height 1 prime if and only if $J = x_1^e$ for some $e > 0$.*

Proof. Since $S(n)$ is a UFD, a height 1 prime is just a principle ideal generated by an irreducible ideal f. Now for general linear change of coordinate α, $\mathrm{in}(\alpha f) = x_1^{deg(f)}$. Conversely for any degree d, we have a polynomial $x_1^d - x_2^{d-1}x_3$. Apply the Eisenstein criterion for the ideal (x_3) we know it is irreducible, and its generic initial monomial is just x_1^d. □

Corollary 6.2. *Let $J = \mathrm{gin}(P)$. Suppose $\mathrm{ht}(J) = 1$, then $J = x_1^e$ for some $e > 0$.*

Proof. $\dim(S/P) = \dim(S/J) = n-1$. So $\mathrm{ht}(P) = 1$ because $S(n)$ is catenary and P is a prime. The rest follows by proposition 6.1.

If we assume $\dim(S/P) = n-d$ in the previous theorems we will get some interesting results. In this case we have the following property: □

Proposition 6.3. *Let $\dim(S/P) = n-d$, $J = \mathrm{gin}(P)$, then:*

1. $x_1,\dots,x_d \in \sqrt{J}$ *and* $x_{d+1},\dots,x_n \notin \sqrt{J}$.
2. $R = k[x_{d+1},\dots,x_n]$ *is a Noether normalization of* S/P *and* S/J.
3. $S/\Phi_d(J)$ *is free over* R *and* $\Phi_d(J)/J$ *is* R*-torsion.*
4. $\deg(S/P) = \deg(S/J) = \mathrm{rank}_R(S/J) = \mathrm{rank}_R(S/\Phi_d(J)) = \dim_k(S/\Phi_d(J) + (x_{d+1}, x_{d+2},\dots,x_n))$.

Now in this case, $r-n+d = 0$, so by lemma 5.2 and lemma 5.3, $\overline{\Phi_d}(J) = \Pi_d(J)+u$ if and only if $\deg(S/\Phi_d(J)) + 1 = \deg(S(d)/\Pi_d(J))$. We can say in this case we have exactly one more degree after applying Φ_d. So by theorem 4.1 we have:

Theorem 6.4. *Let P be a prime ideal in $S = S(n)$, $\dim(S/P) = n-d$, $J = \operatorname{gin}(P)$, and J satisfies $\deg(S/\Phi_d(J)) + 1 = \deg(S(d)/\Pi_d(J))$. Then J is generated by monomials in $k[x_1, \dots, x_{d+1}]$ and x_{d+1} appears in the minimal generator of J. Moreover, $\operatorname{depth}(S/P) = n-d-1$ and S/P is almost Cohen-Macaulay.*

Remark 6.5. This is a generalization of a lemma in Kwak's paper [2], Theorem 5.1 where we have $S(d)/\Pi_d(J) = S(d)/\mathfrak{m}(d)^{r+1}$. Here r is the reduction number of S/P.

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